Announcements

1) New HW up, due next Friday
Also has a webwork and
a written portion (on CTools
under "Assignments")

2) No class Monday

3) Drop / Add - last day
is today

$Ax = b, \quad b \neq \vec{0}$

Any solution to $Ax = b$

can be written as

$\boxed{x_0 + a y_0 = x}$ where

x_0 is a particular solution

to $Ax = b$ and $Ay_0 = \vec{0}$.

This is true since

$$Ax = A(x_0 + ay_0)$$

$$= Ax_0 + A(ay_0)$$

$$= \underbrace{Ax_0}_{= b} + a\underbrace{Ay_0}_{= \vec{0}}$$

$$= b$$

Example 1: Solving

$$2x - 3y + z = 1$$
$$x + 4y - 6z = 0$$

Find one solution: Let $x = 0$.

Then we get

$$-3y + z = 1$$
$$4y - 6z = 0$$

$$\to \begin{bmatrix} -3 & 1 & 1 \\ 4 & -6 & 0 \end{bmatrix}$$

rref

$$\begin{bmatrix} 1 & 0 & -3/7 \\ 0 & 1 & -2/7 \end{bmatrix}$$

This says $\begin{bmatrix} 0 \\ -3/7 \\ -2/7 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

is a solution to our system.

To get all solutions, we solve the homogeneous system

$$2x - 3y + z = 0$$

$$x + 4y - 6z = 0$$

If $x=1$ (choose a nonzero x), we get

$$2-3y+z=0$$
$$1+4y-6z=0$$

$$\rightarrow \quad -3y+z=-2$$
$$4y-6z=-1$$

$$\rightarrow \begin{bmatrix} -3 & 1 & -2 \\ 4 & -6 & -1 \end{bmatrix}$$

$$\text{rref} \begin{bmatrix} 1 & 0 & -11/14 \\ 0 & 1 & -5/14 \end{bmatrix}$$

This says $\begin{bmatrix} 1 \\ -11/14 \\ -5/14 \end{bmatrix}$

is a solution to the homogeneous equation, so any solution looks like

$$\boxed{\begin{bmatrix} 0 \\ -3/7 \\ -2/7 \end{bmatrix} + a \begin{bmatrix} 1 \\ -11/14 \\ -5/14 \end{bmatrix}}$$

How to tell if one vector is in the span of others

v is in the span of $v_1, v_2, \ldots, v_n$ if the matrix equation

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = v$$

same

has a solution

Example 2: Is

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ in the span of}$$

$$\underbrace{\left\{ \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \right\}}_{\text{notation for a set of vectors}} ?$$

We're solving

$$\begin{bmatrix} -1 & 3 \\ 1 & 0 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 & 1 \\ 1 & 0 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$ is our

matrix, row reduce

rref

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inconsistent, so $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is not

in the span of $\begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$

If the rref gave solutions, then the vector v would be in the span of $v_1, v_2, \ldots, v_n$.

Example 3:

Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the

Span of $\left\{ \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}$?

Matrix $\begin{bmatrix} 6 & 0 & 1 \\ 0 & 2 & 2 \\ 6 & 3 & 3 \end{bmatrix}$

rref $\begin{bmatrix} 1 & 0 & 1/6 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

This says

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in the span

of $\left\{ \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}$ since

the rref was not inconsistent.

The solution is $x = 1/6$, $y = 1$, so

$$\boxed{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1/6 \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}}$$

Linear Independence

This, along with span, is one of the first crucial definitions in this class.

v is linearly independent of $v_1, v_2, \dots, v_n$ if v is not in the span of $v_1, v_2, \dots, v_n$.

Written as v not in $\text{span}\{v_1, v_2, \dots, v_n\}$.

Example 4: We saw

that $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is not in

$$\operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

in Example 2.

This means $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is

linearly independent from

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

Alternate Characterization

A vector v is linearly independent from $v_1, v_2, \dots, v_n$ if whenever

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n + a v = \vec{0},$$

we have $a_1 = a_2 = \cdots = a_n = a = 0$ for numbers $a_1, a_2, \dots, a_n, a$.